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First-order supersymmetric sigma models and target space geometry

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Abstract: We study the conditions under which $N = (1,1)$ generalized sigma models support an extension to $N = (2,2)$. The enhanced supersymmetry is related to the target space complex geometry. Concentrating on a simple situation, related to Poisson sigma models we develop a language that may help us analyze more complicated models in the future. In particular, we uncover a geometrical framework which contains generalized complex geometry as a special case.

Keywords: sigma model, supersymmetry, generalized complex geometry.

Contents

1. Introduction

Supersymmetry has a number of interesting relations to geometry. The analogue of Minkowski space is superspace, whose geometry is nontrivial even in the 'flat' case [1]. Curved superspace is the setting for supergravity and has a wealth of interesting geometrical aspects [1, 2, 3]. Superembeddings in curved superspace constrains the geometry very stringently and in many cases even determines the dynamics of the embedded super p-branes [4]. Extended supersymmetry is covariantly described in various extended superspaces with auxiliary degrees of freedom [5, 6, 7]. The target space of supersymmetric nonlinear sigma models, finally, has to be of a certain type depending on the dimension and on the number of supersymmetries. It is this latter situation which concerns us in this paper, more precicely the geometry of twodimensional $N = (2,2)$ supersymmetric nonlinear sigma models with an antisymmetric B-field.

In a classic paper [8] it was shown that the target space of such a sigma model has to be bi-hermitean, i.e. there are two complex structures preserving the metric and they are covariantly constant with respect to connections whose torsions are $\pm \mathrm{d}B$. Recently this geometry has been reinterpreted in terms of a generalized complex geometry, which arose in the context of generalized Calabi-Yau manifolds with B-field fluxes [9]. In [10] many aspects of this geometry are investigated and described. In particular, it is shown that a subclass called generalized Kähler geometry precicely describes the bi-hermitean geometry.

A natural question to ask is then how generalized Kähler geometry can be directly realized in a sigma model. Since generalized complex geometry is defined on the sum of the tangent and cotangent bundles, $TM \oplus T^*M$, and the usual sigma model is defined only on TM, the first task is to find an appropriate extension of the sigma model to include fields on T^*M. This was done in [11] where auxiliary spinorial T^*M-fields were introduced in the $N = (1,1)$ model and the conditions for non-manifest $N = (2,2)$ supersymmetry investigated under certain assumptions. This investigation was repeated in [12], for the case when the metric is absent. Relaxing these assumptions and limiting the study mainly to extending $N = (1,0)$ to $N = (2,0)$, a direct relation to generalized complex geometry was found in in most cases [13]. However, in that investigation it seemed that the geometry in the $N = (2,0)$ case might be even more general, although the study was incomplete.

To further investigate the geometry, a manifest $N = (2,2)$ model in terms of left and right (anti-)chiral superfields [14] was reduced to $N = (1,1)$ superfields and the generalized complex structures identified in [15]. An interesting aspect of this model is that the reduction automatically provides the auxiliary spinorial $N = (1,1)$ fields.

In a separate line of investigation [16, 17], it has been shown that generalized complex geometry bears a close relation to the Batalin-Vilkovisky (BV) treatment of the Poisson sigma model, or more precisely to the Hitchin sigma model. Namely, the generalized complex geometry implies that the BV-master equation is satisfied. Also in this case the implication seems to go only in one direction. Generalized complex geometry has also appeared in the sigma model context, e.g. in a hamiltonian discussion [18] and for topological strings [19].

The reason that the investigation of the conditions for $N = (2,0)$ supersymmetry (and for $N = (2,2)$ supersymmetry) was not carried out [13] was mainly the technical complications of having to find solutions to a large number of algebraic and differential constraints. In the present paper we show how an appropriate field-redefinition can be used to put the sigma model action in a form where invariance under supersymmetry restricts many of the tensors in the supersymmetry transformations of the fields to vanish. This allows us to completely determine the target space geometry, at least for the case of vanishing metric, i.e. with only a B-field present.

In doing this we unravel a target space structure where the natural objects live on $TM \oplus (T^*M_+ \oplus T^*M_-)$, i.e. the geometry involves two copies of the cotangent bundle rather than one. Correspondingly all the fundamental geometric objects such as almost complex structures, metric and connections have a natural formulation in terms of $3d \times 3d$ matrices. In some respects this structure resembles the bi-hermitean geometry of the second order action (auxiliary fields removed) more than the generalized Kähler geometry. In particular, the Courant integrability condition of the generalized complex geometry is replaced by covariantly constancy of the matrix-valued almost complex structures. Now, one of the nice features of generalized complex geometry is that it naturally puts the so-called b-transform on the same footing as the diffeomorphisms since they are both automorphisms of the Courant-bracket. It is thus gratifying that we find that the b-transform can be extended to act on our matrix-objects, and that this extended b-transform is indeed a gauge transformation of our basic bundle which preserves the covariantly constancy condition. Finally, under certain conditions the $3d \times 3d$ matrices collapse to $2d \times 2d$ matrices recovering generalized complex geometry. In other words, the latter is contained in the structure we have found.

The paper is organized as follows: After a short recapitulation of the basic facts about $N = (2, 2)$ supersymmetric sigma models in section 2, we turn towards a toy model which we extend to a first order formalism in section 3. For this model, we give a huge family of solutions for the additional supersymmetry that all close off-shell. Section 4 is devoted to the development of a proper language that collects the results in a way similar to the notion of generalized complex geometry. Based on these results, we discuss in section 5 how to find more general solutions. In section 6, we show how this relates to the geometry of $N = (2, 2)$ symplectic sigma models in a way that extends the b-transformation. In section 7 we speculate about the role of manifest $N = (2, 2)$ supersymmetry before ending with a short discussion and open questions in section 8.

2. $N = (2, 2)$ sigma models, preliminaries

The action for a $N = (1, 1)$ supersymmetric non-linear sigma model under the presence of a background metric $G_{\mu\nu}$ and an antisymmetric field $B_{\mu\nu}$

$$S = \int \mathrm{d}^2\xi \mathrm{d}^2\theta D_+\phi^\mu E_{\mu\nu}(\phi) D_-\phi^\nu \tag{2.1}$$

possesses $N = (2, 2)$ supersymmetry [8] provided that the target space geometry is bi-hermitian. Here, $D_\pm$ are the spinorial derivatives, $D_\pm^2 = \mathrm{i}\partial_{\substack{+\\=}}$, and $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$. The additional, non-manifest supersymmetry is given by

$$\delta\phi^\mu = \epsilon^+ J^{(+)\mu}_{\nu} D_+\phi^\nu + \epsilon^- J^{(-)\mu}_{\nu} D_-\phi^\nu \tag{2.2}$$

where $J^{(\pm)}$ are complex structures. The metric is hermitian with respect to both of them and the complex structures are covariantly constant, i.e.

$$\begin{aligned} &J^{(\pm)2} = -\mathbb{1} && N(J^{(\pm)}) = 0 \\ &G_{\mu\nu} = J^{(\pm)\kappa}_{\mu} G_{\kappa\lambda} J^{(\pm)\lambda}_{\nu} && \nabla^{(\pm)}_{\rho} J^{(\pm)\mu}_{\nu} = 0. \end{aligned} \tag{2.3}$$

Here, $N(J^{(\pm)})$ is the Nijenhuis torsion for $J^{(\pm)}$,

$$N(J^{(\pm)})^{\mu}_{\alpha\beta} = J^{(\pm)\mu}_{\rho} J^{(\pm)\rho}_{[\beta\alpha]} - J^{(\pm)\rho}_{[\alpha} J^{(\pm)\mu}_{\beta]\rho}. \tag{2.4}$$

The covariant derivatives $\nabla^{(\pm)}$ are given by the connections

$$\Gamma^{(\pm)\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} \pm T^{\alpha}_{\beta\gamma} \tag{2.5}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ is the metric connection and $T^{\alpha}_{\beta\gamma} = \frac{1}{2} H_{\beta\gamma\kappa} G^{\kappa\alpha}$ is the torsion. This implies that $H = \mathrm{d}B$ is related to the complex structures in a certain way.

The above conditions ensure that the additional supersymmetry commutes with the first manifest supersymmetry and that its algebra closes on-shell. Off-shell closure is achieved provided that the two complex structures commute,

$$[J^{(+)}, J^{(-)}] = 0. \tag{2.6}$$

This and (2.3) imply that the Magri-Morosi concomitant [20, 21]

$$M(J^{(+)}, J^{(-)})^{\mu}_{\alpha\beta} = J^{(+)\mu}_{\alpha\rho} J^{(-)\rho}_{\beta} - J^{(-)\mu}_{\beta\rho} J^{(+)\rho}_{\alpha} + J^{(+)\mu}_{\rho} J^{(-)\rho}_{\alpha\beta} - J^{(-)\mu}_{\rho} J^{(+)\rho}_{\beta\alpha} \tag{2.7}$$

vanishes and that both complex structures and the product structure $\pi = J^{(+)} J^{(-)}$ are integrable and simultaneously diagonalizable.

While in the previous discussion the metric $G_{\mu\nu}$ played a crucial role, we now repeat the analysis in the case of an antisymmetric background field $B_{\mu\nu}$ only, i.e. we set $E_{\mu\nu} \equiv B_{\mu\nu}$ in the action (2.1) and obtain

$$S_B = \int \mathrm{d}^2\xi \mathrm{d}^2\theta D_+ \phi^{\mu} B_{\mu\nu}(\phi) D_- \phi^{\nu}. \tag{2.8}$$

Requiring off-shell supersymmetry, we learn that the set of constraints on the transformations (2.2) reduces to

$$\begin{aligned} &J^{(\pm)2} = -1 && N(J^{(\pm)}) = 0 && H = 0 \\ &[J^{(+)}, J^{(-)}] = 0 && M(J^{(+)}, J^{(-)}) = 0. \end{aligned} \tag{2.9}$$

Thus, the target-space geometry is bicomplex. The condition $H = 0$ implies that the model is topological. This is a perfect toy model for our purpose, as we see it as a first step towards understanding more general sigma models with extended supersymmetry.

3. Auxiliary fields and supersymmetry algebra

First order sigma model actions have recently come into the focus of research due to their relation to generalized complex geometry on the target manifold. While it is straightforward but lengthy to work out the on-shell supersymmetry transformations [11], off-shell supersymmetry is still not really understood in geometrical terms, partly due to the lack of notation. Several attempts were made to identify those models that admit or require generalized complex geometry [12, 13, 16, 17, 18, 22, 23]. Here, we follow a different approach to investigate the question of off-shell supersymmetry. We focus on the action (2.8) and introduce spinorial auxiliary fields $S_\pm$ on T^*M. They are combined into an auxiliary term added to the action

$$S = \int \mathrm{d}^2\xi \mathrm{d}^2\theta \Big[S_{+\mu} \Pi^{\mu\nu} S_{-\nu} + D_+\phi^\mu B_{\mu\nu} D_-\phi^\nu \Big]. \tag{3.1}$$

To keep things simple, we assume that Π is a Poisson tensor of full rank, i.e. it is symplectic and hence satisfies the Jacobi identity $\Pi^{[\alpha\beta}{}_{\rho}\Pi^{\rho|\gamma]} = 0$.

By dimensional arguments, see e.g. [11], the most general form of the second supersymmetry is given by

$$\begin{aligned}
\delta^{(\pm)}\phi^\mu =& \epsilon^\pm \left(D_\pm\phi^\nu J_\nu^{(\pm)\mu} - S_{\pm\nu} P^{(\pm)\mu\nu} \right) \\
\delta^{(\pm)} S_{\pm\mu} =& \epsilon^\pm \big(D_\pm^2 \phi^\nu L^{(\pm)}_{\mu\nu} - D_\pm S_{\pm\nu} K_\mu^{(\pm)\nu} + S_{\pm\nu} S_{\pm\sigma} N_\mu^{(\pm)\nu\sigma} \\
& \qquad + D_\pm\phi^\nu D_\pm\phi^\rho M^{(\pm)}_{\mu\nu\rho} + D_\pm\phi^\nu S_{\pm\sigma} Q^{(\pm)\sigma}_{\mu\nu} \big) \\
\delta^{(\pm)} S_{\mp\mu} =& \epsilon^\pm \big(D_\pm S_{\mp\nu} R_\mu^{(\pm)\nu} + D_\mp S_{\pm\nu} Z_\mu^{(\pm)\nu} + D_\pm D_\mp \phi^\nu T^{(\pm)}_{\mu\nu} \\
& \qquad + S_{\pm\rho} D_\mp\phi^\nu U^{(\pm)\rho}_{\mu\nu} + D_\pm\phi^\nu S_{\mp\rho} V^{(\pm)\rho}_{\mu\nu} \\
& \qquad + D_\pm\phi^\nu D_\mp\phi^\rho X^{(\pm)}_{\mu\nu\rho} + S_{\pm\nu} S_{\mp\rho} Y_\mu^{(\pm)\nu\rho} \big) .
\end{aligned} \tag{3.2}$$

The action (3.1) is invariant under these transformations provided that

$$\Pi^{\mu\alpha} R^\beta_\alpha = -K^\mu_\nu \Pi^{\nu\beta} \qquad \Pi^{(\alpha|\rho} Z^{\beta)}_\rho = 0 \qquad L_{\alpha\beta} = 0 \qquad T_{\alpha\beta} = 0 \tag{3.3}$$

and that a set of differential equations hold. One of these is $H = 0$. For the time being, we make the assumption that $P^{(+)}$ and $P^{(-)}$ are invertible. It turns out that things simplify drastically under this assumption. Indeed, already the commutators of the second supersymmetry with itself provide 113 conditions to be satisfied. We comment on the situation for more general $P^{(\pm)}$ in section 5. Off-shell closure of the additional supersymmetry algebra is guaranteed if $J^{(\pm)}$ are commuting complex structures that are covariantly constant with respect to certain torsionfree connections $\Gamma^{(J^{(\pm)})\mu}_{\nu\rho}$

$$J^{(\pm)2} = -1 \qquad [J^{(+)}, J^{(-)}] = 0 \qquad \nabla^{(J^{(\pm)})} J^{(\pm)} = 0 \tag{3.4}$$

The transformations (3.2) are determined by the composite tensors:

$$
\begin{aligned}
K_\alpha^{(\pm)\beta} &= -P_{\alpha\mu}^{(\pm)} J_\nu^{(\pm)\mu} P^{(\pm)\nu\beta} \\
R_\alpha^{(\pm)\beta} &= -\Pi_{\alpha\mu} K_\nu^{(\pm)\mu} \Pi^{\nu\beta} \\
L_{\alpha\beta}^{(\pm)} &= 0 \\
T_{\alpha\beta}^{(\pm)} &= 0 \\
Z_\beta^{(\pm)\alpha} &= -P_{\beta\kappa}^{(\mp)} P^{(\pm)\kappa\lambda} R_\lambda^{(\mp)\alpha} + P_{\beta\kappa}^{(\mp)} J_\lambda^{(\mp)\kappa} P^{(\pm)\lambda\alpha}
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
M_{\mu\nu\rho}^{(\pm)} &= 0 \\
X_{\mu\nu\rho}^{(\pm)} &= 0 \\
Q_{\mu\nu}^{(\pm)\rho} &= \Gamma_{\beta\mu}^{(K^{(\pm)})\rho} J_\nu^{(\pm)\beta} + \Gamma_{\nu\kappa}^{(K^{(\pm)})\rho} K_\mu^{(\pm)\kappa} \\
V_{\mu\gamma}^{(\pm)\beta} &= \Gamma_{\rho\mu}^{(R^{(\pm)})\beta} J_\gamma^{(\pm)\rho} - \Gamma_{\gamma\rho}^{(R^{(\pm)})\beta} R_\mu^{(\pm)\rho} \\
U_{\beta\gamma}^{(\pm)\alpha} &= \Gamma_{\gamma\kappa}^{(K^{(\pm)})\alpha} Z_\beta^{(\pm)\kappa} \\
N_\gamma^{(\pm)[\alpha\beta]} &= \Gamma_{\kappa\gamma}^{(K^{(\pm)})[\alpha} P^{(\pm)\kappa|\beta]} \\
Y_\gamma^{(\pm)\alpha\beta} &= -\Gamma_{\rho\gamma}^{(R^{(\pm)})\beta} P^{(\pm)\rho\alpha},
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
\Gamma_{\rho\lambda}^{(K^{(\pm)})\sigma} &= \left[P_{\lambda\mu\rho}^{(\pm)} - P_{\lambda\nu}^{(\pm)} \Gamma_{\rho\mu}^{(J^{(\pm)})\nu}\right] P^{(\pm)\mu,\sigma} \\
\Gamma_{\rho\lambda}^{(R^{(\pm)})\sigma} &= \left[\Pi_{\lambda\mu,\rho} - \Pi_{\lambda\nu} \Gamma_{\rho\mu}^{(K^{(\pm)})\nu}\right] \Pi^{\mu\sigma}.
\end{aligned}
\tag{3.7}
$$

The second rank tensors are 'covariantly constant' according to

$$
\begin{aligned}
\nabla_\rho J_\beta^{(\pm)\alpha} &\equiv J_{\beta,\rho}^{(\pm)\alpha} - \Gamma_{\rho\beta}^{(J^{(\pm)})\nu} J_\nu^{(\pm)\alpha} + \Gamma_{\rho\nu}^{(J^{(\pm)})\alpha} J_\beta^{(\pm)\nu} = 0 \\
\nabla_\rho K_\beta^{(\pm)\alpha} &\equiv K_{\beta,\rho}^{(\pm)\alpha} - \Gamma_{\rho\beta}^{(K^{(\pm)})\nu} K_\nu^{(\pm)\alpha} + \Gamma_{\rho\nu}^{(K^{(\pm)})\alpha} K_\beta^{(\pm)\nu} = 0 \\
\nabla_\rho R_\beta^{(\pm)\alpha} &\equiv R_{\beta,\rho}^{(\pm)\alpha} - \Gamma_{\rho\beta}^{(R^{(\pm)})\nu} R_\nu^{(\pm)\alpha} + \Gamma_{\rho\nu}^{(R^{(\pm)})\alpha} R_\beta^{(\pm)\nu} = 0 \\
\nabla_\rho P^{(\pm)\alpha\beta} &\equiv P^{(\pm)\alpha\beta}{}_{,\rho} + P^{(\pm)\alpha\nu} \Gamma_{\rho\nu}^{(K^{(\pm)})\beta} + \Gamma_{\rho\nu}^{(J^{(\pm)})\alpha} P^{(\pm)\nu\beta} = 0 \\
\nabla_\rho Z_\beta^{(\pm)\alpha} &\equiv Z_{\beta,\rho}^{(\pm)\alpha} - \Gamma_{\rho\beta}^{(R^{(\pm)})\nu} Z_\nu^{(\pm)\alpha} + \Gamma_{\rho\nu}^{(K^{(\pm)})\alpha} Z_\beta^{(\pm)\nu} = 0.
\end{aligned}
\tag{3.8}
$$

The connections are related as

$$
\Gamma^{(J^{(-)})} = \Gamma^{(J^{(+)})} \qquad\qquad \Gamma^{(R^{(\pm)})} = \Gamma^{(K^{(\mp)})}.
\tag{3.9}
$$

The corresponding Riemann tensors $R^{(\cdot)\kappa}{}_{\lambda\mu\nu} = \Gamma^{(\cdot)\kappa}_{[\nu|\lambda,|\mu]} + \Gamma^{(\cdot)\rho}_{[\nu|\lambda}\Gamma^{(\cdot)\kappa}_{\mu]\rho}$ vanish:

$$
R^{(R^{(\pm)})} = R^{(K^{(\pm)})} = R^{(J^{(\pm)})} = 0.
\tag{3.10}
$$

From the non-derivative parts of the algebra, one constraint remains:

$$
R_\mu^{(\pm)\rho} Z_\rho^{(\pm)\alpha} + Z_\mu^{(\pm)\rho} K_\rho^{(\pm)\alpha} = 0.
\tag{3.11}
$$

We observe that, except for being covariantly constant, there is no constraint on $P^{(\pm)}$. Equations (3.5) imply

$$Z_{\beta}^{(\pm)\alpha} = P_{\beta\kappa}^{(\mp)}[J^{(\mp)}, P^{(-)}\Pi^{-1}P^{(+)t}]^{\kappa\alpha}. \tag{3.12}$$

The relation (3.12) shows that it is possible, at least in certain situations, to choose $P^{(\pm)}$ in such a way that both $Z^{(\pm)}$ vanish. This requires both complex structures to commute with $\omega^{\alpha\beta} \equiv (P^{(-)}\Pi^{-1}P^{(+)t})^{\alpha\beta} = -(P^{(+)}\Pi^{-1}P^{(-)t})^{\beta\alpha}$. In other words, ω has to be antihermitian with respect to both complex structures. If, on the other hand, ω is antisymmetric and in additions satisfies the Jacobi identity then we may identify its inverse with the two-form of a symplectic manifold. Clearly, $G^{(\pm)} = J^{(\pm)}\omega$ are then candidates for effective metrics. One such example is the case $P^{(-)\alpha\beta} = P^{(+)\alpha\beta}$. It follows that $R^{(\pm)} = -K^{(\pm)}$, $[K^{(+)}, K^{(-)}] = 0$ and Π is antihermitian with respect to $K^{(\pm)}$. However, this alternative is only possible if Π is covariantly constant.

$$\nabla_{\rho}^{(K)}\Pi^{\alpha\beta} = \Pi^{\alpha\beta}{}_{,\rho} + \Gamma_{\rho\kappa}^{(K)[\alpha}\Pi^{\kappa|\beta]} = 0, \tag{3.13}$$

where $\Gamma^{(K)} \equiv \Gamma^{(K^{(+)})} = \Gamma^{(K^{(-)})}$.

This covers the discussion of the second supersymmetry transformations under the assumptions (3.4) for the particular model we study. Equation (3.9) is sufficient for off-shell closure. It might not be necessary though we find this quite unlikely due to the way (3.9) contributes to the solution.

4. Almost complex structures on $TM \oplus (T^*M_+ \oplus T^*M_-)$

In the previous section we found that the complete data identifying the solution is encoded in the objects B, Π, $J^{(\pm)}$, $P^{(\pm)}$ and $\Gamma^{(J)}$. We want a formulation as closely related as possible to generalized complex geometry [9, 10] and shall try to find a role for the components of (3.2) in that context. We start with a recapitulation of the notion of generalized complex geometry.

An almost complex structure is a linear map $J : TM \to TM$ that squares to -1. If we define projection operators $\pi_{\pm} = \frac{1}{2}(1 \pm \mathrm{i}J)$, then J is integrable if

$$\pi_{\mp}[\pi_{\pm}X, \pi_{\pm}Y] = 0 \tag{4.1}$$

for any $X, Y \in TM$, where $[\cdot,\cdot]$ is the Lie bracket on TM. Hitchin [9] proposed and later Gualtieri [10] investigated a generalization of this notion, where TM is replaced by $TM \oplus T^*M$ and the Lie bracket is replaced by the so-called Courant bracket. A generalized complex structure is defined as a map $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$, such that $\mathcal{J}^2 = -1$ and it leaves the natural symmetric inner product

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(i_X\eta + i_Y\xi) \qquad X + \xi, Y + \eta \in TM \oplus T^*M \tag{4.2}$$

invariant. In a coordinate basis $(\partial_\mu, \mathrm{d}x^\mu)$, the metric

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

is hermitian with respect to $\mathcal{J}$. Furthermore, the $+\mathrm{i}$ eigenbundle of $\mathcal{J}$ is closed under the Courant bracket [24], which is defined as

$$[X+\xi, Y+\eta]_C = [X,Y] + L_X\eta - L_Y\xi - \tfrac{1}{2}\mathrm{d}(i_X\eta - i_Y\xi). \tag{4.4}$$

This bracket allows to define Courant integrability as a straightforward generalization of (4.1). In a coordinate basis, generalized complex structures can be written in terms of $2d \times 2d$ matrices

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}. \tag{4.5}$$

An important feature of the Courant bracket is the existence of non-trivial automorphisms defined by closed two-forms $b \in \Omega^2_{\text{closed}}(M)$. Consequently, given a generalized complex structure $\mathcal{J}$, we can define a new such structure by the b-transformation

$$\mathcal{J}_b = \mathcal{U}\mathcal{J}\mathcal{U}^{-1} \qquad\qquad \mathcal{U} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}. \tag{4.6}$$

The automorphism of the Courant bracket guarantees this structure to be integrable. For a detailed discussion, we refer to the original works [9, 10].

In [13], the authors constructed examples of sigma models admitting generalized complex geometry in the target space. Mainly as a curiosity, they found that the algebraic conditions for closure of the algebra could be combined into a single $3d \times 3d$ matrix squaring to -1. This object seems like a natural extension of the concept of generalized complex structures. Here, we elaborate this idea in detail and use it as a basis for the description of the target space geometry. We thus combine the tensors into two $3d \times 3d$ matrices

$$\boldsymbol{J}^{(+)} = \begin{pmatrix} J^{(+)} & -P^{(+)} & 0 \\ -L^{(+)} & K^{(+)} & 0 \\ T^{(+)} & -Z^{(+)} & R^{(+)} \end{pmatrix} \qquad \boldsymbol{J}^{(-)} = \begin{pmatrix} J^{(-)} & 0 & -P^{(-)} \\ T^{(-)} & R^{(-)} & -Z^{(-)} \\ -L^{(-)} & 0 & K^{(-)} \end{pmatrix}. \tag{4.7}$$

The components of these matrices are the linear maps

$$\begin{aligned} &J^{(+)} : TM \to TM && P^{(+)} : T^*M_+ \to TM \\ &L^{(+)} : TM \to T^*M_+ && K^{(+)} : T^*M_+ \to T^*M_+ \\ &T^{(+)} : TM \to T^*M_- && Z^{(+)} : T^*M_+ \to T^*M_- \quad R^{(+)} : T^*M_- \to T^*M_-. \end{aligned} \tag{4.8}$$

The components of $\boldsymbol{J}^{(-)}$ are defined analogously. Here, T^*M_+ and T^*M_- are two copies of the cotangent bundle. They are associated with the two Grassmann directions on the worldsheet. Thus, $\boldsymbol{J}^{(\pm)}$ map the bundle $E = TM \oplus (T^*M_+ \oplus T^*M_-)$ onto itself. Guided by the action (3.1) we introduce a (degenerate) symmetric inner product on E, an equivalent to the metric for the ordinary sigma model:

$$\boldsymbol{G} = \boldsymbol{G}^t = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Pi \\ 0 & \Pi^t & 0 \end{pmatrix}. \tag{4.9}$$

We note that $\boldsymbol{G}$ is degenerate because we set $E_{(\mu\nu)} = 0$ in (2.8) and that $\boldsymbol{G}$ is antisymmetric in the fermionic components. The algebraic conditions arising from the invariance of the action, eqns. (3.3), and the non-differential part of the algebra (3.5) can be written in a compact way:

$$\boldsymbol{J}^{(\pm)t}\boldsymbol{G}\boldsymbol{J}^{(\pm)} = \boldsymbol{G} \qquad \boldsymbol{J}^{(\pm)2} = -\mathbf{1} \qquad [\boldsymbol{J}^{(+)}, \boldsymbol{J}^{(-)}] = 0. \tag{4.10}$$

This allows us to regard $\boldsymbol{J}^{(\pm)}$ as (almost) complex structures on E. Eqns. (3.8) tell us that these structures are covariantly constant,

$$\boldsymbol{\nabla}^{(\pm)}\boldsymbol{J}^{(\pm)} \equiv \boldsymbol{\partial}\boldsymbol{J}^{(\pm)} - \boldsymbol{J}^{(\pm)} \cdot \boldsymbol{\Gamma}^{(\pm)} + \boldsymbol{\Gamma}^{(\pm)} \cdot \boldsymbol{J}^{(\pm)} = 0 \tag{4.11}$$

with respect to certain connection matrices

$$\begin{aligned} \boldsymbol{\Gamma}^{(+)} &= \operatorname{diag}\left(\Gamma^{(J^{(+)})}, -\Gamma^{(K^{(+)})}, -\Gamma^{(R^{(+)})}\right) \\ \boldsymbol{\Gamma}^{(-)} &= \operatorname{diag}\left(\Gamma^{(J^{(-)})}, -\Gamma^{(R^{(-)})}, -\Gamma^{(K^{(-)})}\right) \end{aligned} \tag{4.12}$$

and a partial derivative $\boldsymbol{\partial} = \mathbf{1}\partial$. Equation (3.9) translates into

$$\boldsymbol{\Gamma} \equiv \boldsymbol{\Gamma}^{(+)} = \boldsymbol{\Gamma}^{(-)}. \tag{4.13}$$

The components of $\boldsymbol{\Gamma}$ are torsionfree, $\boldsymbol{\Gamma}^t = \boldsymbol{\Gamma}$, where the transposition is acting on the two lower indices, and its Riemann tensor is

$$\boldsymbol{R} = [\boldsymbol{\nabla}, \boldsymbol{\nabla}] = \mathbf{d}\boldsymbol{\Gamma} - \boldsymbol{\Gamma} \circ \boldsymbol{\Gamma} \tag{4.14}$$

where $\mathbf{d} = \mathbf{1}\mathrm{d}$ is the generalized exterior derivative. According to (3.10), this matrix vanishes:

$$\boldsymbol{R} = 0. \tag{4.15}$$

In Kähler geometry, the Nijenhuis torsion and the Levi-Civita connection are related by

$$N(J)(X,Y) = (\nabla_{JX}J)Y - (\nabla_{JY}J)X + (\nabla_X J)JY - (\nabla_Y J)JX, \tag{4.16}$$

with $X, Y \in TM$. Clearly, if J is covariantly constant with respect to the Levi-Civita connection, then $N(J) = 0$. The generalization to a matrix-valued Nijenhuis torsion $\boldsymbol{N}(\boldsymbol{J}^{(\pm)})$ would make use of $\boldsymbol{\nabla}$ and $\boldsymbol{\Gamma}$ and hence vanishes if $\boldsymbol{\nabla}\boldsymbol{J}^{(\pm)} = 0$. Thus, (4.11) is an integrability condition ensuring the integrability of $J^{(\pm)}$, $K^{(\pm)}$ and $R^{(\pm)}$.

We find that the above description completely covers closure of the supersymmetry algebra and most of the conditions that arise from the invariance of the action. In fact, the only condition left is $H = 0$. We define an antisymmetric tensor by

$$\boldsymbol{B} = \frac{1}{2}\begin{pmatrix} 2B & 0 & 0 \\ 0 & 0 & \Pi \\ 0 & -\Pi^t & 0 \end{pmatrix} \tag{4.17}$$

and define its field strength in the usual way,

$$\boldsymbol{H} = \mathbf{d}\boldsymbol{B} = \frac{1}{2}\begin{pmatrix} 2H_B & 0 & 0 \\ 0 & 0 & \Pi H_\Pi \Pi \\ 0 & \Pi H_\Pi \Pi & 0 \end{pmatrix}. \tag{4.18}$$

Here, $H_\Pi = \mathrm{d}(\Pi^{-1})$ which vanishes in our case, since Π is symplectic. With this, we have

$$\boldsymbol{H} = 0. \tag{4.19}$$

There are actually four different possibilities for choosing the two almost complex structure matrices describing one and the same situation. They are obtained from (4.7) by acting on $\boldsymbol{J}^{(\pm)}$ with $\boldsymbol{C}^{(\pm)} = \mathrm{diag}(1, \mp 1, \pm 1)$ and $\boldsymbol{S} = \boldsymbol{C}^{(+)}\boldsymbol{C}^{(-)}$:

$$\begin{aligned} \boldsymbol{J}_1^{(\pm)} &= \boldsymbol{J}^{(\pm)} & \qquad \boldsymbol{J}_2^{(\pm)} &= \boldsymbol{C}^{(+)}\boldsymbol{J}_1^{(\pm)}\boldsymbol{C}^{(+)} \\ \boldsymbol{J}_3^{(\pm)} &= \boldsymbol{S}\boldsymbol{J}_1^{(\pm)}\boldsymbol{S} & \qquad \boldsymbol{J}_4^{(\pm)} &= \boldsymbol{C}^{(-)}\boldsymbol{J}_1^{(\pm)}\boldsymbol{C}^{(-)}. \end{aligned} \tag{4.20}$$

The covariant derivative is changed accordingly, e.g.

$$\boldsymbol{\partial}_2 = \boldsymbol{C}^{(+)}\boldsymbol{\partial} \qquad\qquad \boldsymbol{\Gamma}_2 = \boldsymbol{C}^{(+)}\boldsymbol{\Gamma}. \tag{4.21}$$

This symmetry is reminicent of the discrete symmetries of the first order sigma model action discussed in, e.g. [11]. The whole discussion may equally well be formulated in terms of any of these choices.

This completes the discussion of the model (3.1) in this language. However, it is worth noticing that the geometry of the ordinary second order sigma model (2.1) is embedded in this framework in a natural way. It corresponds to

$$\boldsymbol{G} = \mathrm{diag}(G, 0, 0) \qquad\qquad \boldsymbol{B} = \mathrm{diag}(B, 0, 0). \tag{4.22}$$

Of course, then $\boldsymbol{\Gamma}^{(+)}$ and $\boldsymbol{\Gamma}^{(-)}$ are no longer related in the same way, since B generates torsion in the tangent space directions.

5. Towards a more general solution

One of the main ingredients of the solution given in section 3 is the invertibility of $P^{(\pm)}$. This assumption was made because the conditions for the supersymmetry algebra to close simplified drastically. This helped us to introduce the compact notation in the previous section. However, the spacetime geometry turned out to be completely empty, since there is neither a metric nor a three-form field strength. Here, we elaborate the case where $P^{(\pm)}$ may have degeneracies. This implies that the tangent bundle complex structures $J^{(\pm)}$ are no longer related to the cotangent bundle ones $K^{(\pm)}$, $R^{(\pm)}$ in a unique way. The non-differential conditions for invariance of the action and closure of the algebra are still ensured by (4.10)

$$\boldsymbol{J}^{(\pm)2} = -1 \qquad [\boldsymbol{J}^{(+)}, \boldsymbol{J}^{(-)}] = 0 \qquad \boldsymbol{J}^{(\pm)t}\boldsymbol{G}\boldsymbol{J}^{(\pm)} = \boldsymbol{G}. \tag{5.1}$$

We observe that the higher order tensors of the solution (3.6) do not depend on $\Gamma^{(J^{(\pm)})}$ but rather on the connections for $K^{(\pm)}$ and $R^{(\pm)}$. This allows us to go beyond flat space in the following way: To stick as close as possible to the solution given in the previous sections, we start with the assumption that there are two connections $\Gamma^{(R^{(\pm)})}$ such that $R^{(\pm)}$ are two covariantly constant complex structures. With this, the solution on the two copies of the cotangent bundle $T^*M_+ \oplus T^*M_-$ remains the same as before, since

$$\Gamma^{(K)\epsilon}_{\rho\mu} = -\Pi^{\epsilon\nu}\big[\Gamma^{(R)\sigma}_{\rho\nu}\Pi_{\sigma\mu} + \Pi_{\nu\mu,\rho}\big]. \tag{5.2}$$

and since closure of the algebra requires $R^{(R^{(\pm)})} = R^{(K^{(\pm)})} = 0$. In order for the higher order tensors to remain defined as in (3.6), we need the further assumption that there exists $A^{(\pm)\alpha}_{\rho\nu}$ such that

$$P^{(\pm)\alpha\beta}{}_{,\rho} + P^{(\pm)\alpha\nu}\Gamma^{(K^{(\pm)})\beta}_{\rho\nu} = A^{(\pm)\alpha}_{\rho\nu}P^{(\pm)\nu\beta}. \tag{5.3}$$

Together with the equation

$$\nabla^{(K^{(\pm)})}_{\sigma}\big[J^{(\pm)\mu}_{\rho}P^{(\pm)\rho\alpha} + P^{(\pm)\mu\rho}K^{(\pm)\alpha}_{\rho}\big] = 0, \tag{5.4}$$

we read off the connection for the tangent bundle $\Gamma^{(J^{(\pm)})^{\alpha}}_{\rho\nu} = -A^{\alpha}_{\rho\nu}$ and learn that $J^{(\pm)}$ only has to be covariantly constant in the directions where $P^{(\pm)}$ is invertible. On $\ker(P)$, we do no longer get any differential conditions and thus, locally, the tangent space geometry becomes bicomplex. This fits to the original second order sigma model with a B-field only where we obtained a bicomplex geometry and no differential conditions for $J^{(\pm)}$. Especially for $P^{(\pm)\mu\nu} = 0$, we recover this situation, as expected, since the supersymmetry transformation for ϕ^{μ} decouples from the auxiliary fields $S_{\pm\mu}$. Since $R^{(J^{(\pm)})}$ now in general is non-vanishing, we obtain a more involved geometry of the tangent bundle, while the cotangent bundle does not carry any additional geometric structure.

6. Symplectic sigma model and B-transformation

The $N = (2, 2)$ supersymmetric symplectic sigma model action [11, 25]

$$S_{SSM} = \int \mathrm{d}^2\xi \mathrm{d}^2\theta \Big[\hat{S}_{(+\mu} D_{-)}\phi^\mu + \hat{S}_{+\mu}\Pi^{\mu\nu}\hat{S}_{-\nu}\Big] \tag{6.1}$$

is obtained from (3.1) by the transformation

$$S_{\pm\mu} \to \hat{S}_{\pm\mu} = S_{\pm\mu} - \Pi_{\mu\nu}D_\pm\phi^\nu \tag{6.2}$$

and by identifying $B_{\mu\nu} \equiv \Pi_{\mu\nu}$. To be a bit more general, however, we consider the action

$$S_{SSM+B} = S_{SSM} + \int \mathrm{d}^2\xi \mathrm{d}^2\theta \Big[D_+\phi^\mu(B_{\mu\nu} - \Pi_{\mu\nu})D_-\phi^\nu\Big]. \tag{6.3}$$

We notice that if we take $\Pi_{\mu\nu}$ to be a globally defined two-form, this is precisely the action used to discuss the WZW term in [13] with the metric set to zero. By rewriting the transformations (3.2) in terms of ϕ and $\hat{S}_\pm$, we obtain the contributions to the new tensors, which we denote by a hat to distinguish them from the previous results. We omit the $(\pm)$ for a better legibility.

$$\begin{aligned}
\hat{J}^\mu_\nu &= J^\mu_\nu - P^{\mu\rho}\Pi_{\rho\nu} \\
\hat{P}^{\mu\nu} &= P^{\mu\nu} \\
\hat{L}_{\mu\nu} &= \Pi_{\mu\rho}\hat{J}^\rho_\nu - K^\rho_\mu\Pi_{\rho\nu} \\
\hat{K}^\nu_\mu &= K^\nu_\mu + \Pi_{\mu\rho}P^{\rho\nu} \\
\hat{T}_{\mu\nu} &= (R^\rho_\mu - Z^\rho_\mu)\Pi_{\rho\nu} - \Pi_{\mu\rho}\hat{J}^\rho_\nu \\
\hat{Z}^\nu_\mu &= Z^\nu_\mu - \Pi_{\mu\rho}P^{\rho\nu} \\
\hat{R}^\nu_\mu &= R^\nu_\mu
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
\hat{N}^{[\mu\nu]}_\rho &= N^{[\mu\nu]}_\rho \\
\hat{M}_{\mu[\nu\rho]} &= \Pi_{\mu\kappa}\hat{J}^\kappa_{[\rho\nu]} + \Pi_{\mu[\nu|,\kappa}\hat{J}^\kappa_{\rho]} + K^\kappa_\mu\Pi_{\kappa[\nu,\rho]} - N^{[\kappa\lambda]}_\mu\Pi_{\kappa\nu}\Pi_{\lambda\rho} - Q^\kappa_{\mu[\nu}\Pi_{\kappa|\rho]} \\
\hat{Q}^\rho_{\mu\nu} &= Q^\rho_{\mu\nu} - \Pi_{\mu\kappa}P^{\kappa\rho}{}_\nu + N^{[\kappa\rho]}_\mu\Pi_{\kappa\nu} - \Pi_{\mu\nu,\kappa}P^{\kappa\rho} \\
\hat{U}^\rho_{\mu\nu} &= U^\rho_{\mu\nu} + \Pi_{\mu\sigma}P^{\sigma\rho}{}_\nu + \Pi_{\mu\nu,\sigma}P^{\sigma\rho} + Y^{\rho\sigma}_\mu\Pi_{\sigma\nu} \\
\hat{V}^\rho_{\mu\nu} &= V^\rho_{\mu\nu} + Y^{\rho\sigma}_\mu\Pi_{\sigma\nu} \\
\hat{X}_{\mu\nu\rho} &= -\Pi_{\mu\sigma}\hat{J}^\sigma_{\nu\rho} - \Pi_{\mu\rho,\sigma}\hat{J}^\sigma_\nu + (R^\sigma_\mu - Z^\sigma_\mu)\Pi_{\sigma\nu,\rho} \\
&\qquad + U^\sigma_{\mu\rho}\Pi_{\sigma\nu} + V^\sigma_{\mu\nu}\Pi_{\sigma\rho} + Y^{\kappa\lambda}_\mu\Pi_{\kappa\nu}\Pi_{\lambda\rho} \\
\hat{Y}^{\nu\rho}_\mu &= Y^{\nu\rho}_\mu.
\end{aligned} \tag{6.5}$$

The transformation of $\boldsymbol{J}^{(\pm)}$ with components (6.4) can be written in a compact way:

$$\hat{\boldsymbol{J}}^{(\pm)} = \boldsymbol{U}\boldsymbol{J}^{(\pm)}\boldsymbol{U}^{-1} \qquad \boldsymbol{U} = \begin{pmatrix} 1 & 0 & 0 \\ -\Pi^{-1} & 1 & 0 \\ -\Pi^{-1} & 0 & 1 \end{pmatrix}. \tag{6.6}$$

This implies that

$$\begin{aligned} \hat{\boldsymbol{G}} &= (\boldsymbol{U}^{-1})^t\boldsymbol{G}\boldsymbol{U}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & \Pi \\ -1 & -\Pi & 0 \end{pmatrix} \\ \hat{\boldsymbol{B}} &= (\boldsymbol{U}^{-1})^t\boldsymbol{B}\boldsymbol{U}^{-1} = \frac{1}{2}\begin{pmatrix} 2(B-\Pi^{-1}) & -1 & -1 \\ 1 & 0 & \Pi \\ 1 & \Pi & 0 \end{pmatrix}. \end{aligned} \tag{6.7}$$

$\hat{\boldsymbol{G}}$ is hermitian with respect to $\hat{\boldsymbol{J}}^{(\pm)}$, $\hat{\boldsymbol{H}} = 0$ and $\boldsymbol{U}$ is unitary. If we regard (6.2) as a gauge transformation, that is, an automorphism of the bundle E, then $\boldsymbol{\Gamma}$ transforms as a connection and (4.11) is invariant, $\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{J}}^{(\pm)} = \boldsymbol{U}\boldsymbol{\nabla}\boldsymbol{J}^{(\pm)}\boldsymbol{U}^{-1} = 0$. Equations (6.6) and (6.2) extend the b-transform (4.6) of generalized complex geometry to our formulation. Hence, it is suggestive to regard (4.11) as an integrability condition. It is puzzling how to fit in (6.2) in a proper way. Obviously,

$$\hat{\boldsymbol{\Lambda}} \neq \boldsymbol{U}\boldsymbol{\Lambda} \qquad \boldsymbol{\Lambda} = (\phi, S_+, S_-)^t \tag{6.8}$$

due to the derivatives on ϕ. In generalized complex geometry, this problem does not occur, since the fermionic derivative can be included in the definition of $\boldsymbol{\Lambda}$. Here, there are two of them, $D_\pm$, which complicates the situation. To inspect this in more detail, we promote the matrices to operators in the following way:

$$\begin{aligned} \mathbb{U} &= \begin{pmatrix} 1 & 0 & 0 \\ -\Pi^{-1}D_+ & 1 & 0 \\ -\Pi^{-1}D_- & 0 & 1 \end{pmatrix} \qquad \mathbb{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Pi \\ 0 & \Pi^t & 0 \end{pmatrix} \\ \mathbb{B} &= \begin{pmatrix} \overleftarrow{D}_{(+}B\overrightarrow{D}_{-)} & 0 & 0 \\ 0 & 0 & \Pi \\ 0 & -\Pi^t & 0 \end{pmatrix}. \end{aligned} \tag{6.9}$$

Even if B is antisymmetric, the inner product $\langle \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2 \rangle \equiv \boldsymbol{\Lambda}_1^t \mathbb{B} \boldsymbol{\Lambda}_2$ is actually symmetric in $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2$. Accordingly, the almost complex structure matrices become

$$\begin{aligned} \mathbb{J}^{(+)} &= \begin{pmatrix} JD_+ & -P & 0 \\ -LD_+^2 & KD_+ & 0 \\ TD_+D_- & -ZD_- & RD_+ \end{pmatrix} \\ \mathbb{C}^{(+)} &= \boldsymbol{C}^{(+)} = \mathrm{diag}(1,-1,1). \end{aligned} \tag{6.10}$$

We introduce the following object:

$$
\begin{aligned}
\mathbb{Q}^{(+)} &= (\mathbb{Q}^{(+)\phi}, \mathbb{Q}^{(+)S_+}, \mathbb{Q}^{(+)S_-})^t \\
\mathbb{Q}^{(+)\phi} &= 0 \\
\mathbb{Q}^{(+)S_+} &= \begin{pmatrix} \overleftarrow{D}_+ M \overrightarrow{D}_+ & \frac{1}{2}\overleftarrow{D}_+ Q & 0 \\ \frac{1}{2} Q \overrightarrow{D}_+ & N & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\mathbb{Q}^{(+)S_-} &= \begin{pmatrix} \overleftarrow{D}_+ X \overrightarrow{D}_- & 0 & \frac{1}{2}\overleftarrow{D}_+ V \\ 0 & 0 & \frac{1}{2}\overleftarrow{D}_- U \\ \frac{1}{2} V \overrightarrow{D}_+ & \frac{1}{2} U \overrightarrow{D}_- & Y \end{pmatrix}
\end{aligned}
\tag{6.11}
$$

and $\mathbb{J}^{(-)}$, $\mathbb{Q}^{(-)}$, $\mathbb{C}^{(-)}$ defined correspondingly. With this notation the transformation is given by

$$
\hat{\mathbf{\Lambda}} = \mathbb{U}\mathbf{\Lambda} \qquad \hat{\mathbb{G}} = (\mathbb{U}^{-1})^t \mathbb{G} (\mathbb{U}^{-1}) \qquad \hat{\mathbb{B}} = (\mathbb{U}^{-1})^t \mathbb{B} (\mathbb{U}^{-1}). \tag{6.12}
$$

The action (3.1) can be written as

$$
S = \frac{1}{2}\int \mathrm{d}^2\xi \mathrm{d}^2\theta\, \mathbf{\Lambda}^t \mathbb{E} \mathbf{\Lambda} = \frac{1}{2}\int \mathrm{d}^2\xi \mathrm{d}^2\theta\, \hat{\mathbf{\Lambda}}^t \hat{\mathbb{E}} \hat{\mathbf{\Lambda}} \tag{6.13}
$$

where $\mathbb{E} = \mathbb{G} + \mathbb{B}$. The supersymmetry transformations become

$$
\delta^{(\pm)}\mathbf{\Lambda} = \epsilon^{\pm} \mathbb{C}^{(\pm)} \mathbb{J}^{(\pm)} \mathbf{\Lambda} + \epsilon^{\pm} \mathbf{\Lambda}^t \mathbb{Q}^{(\pm)} \mathbf{\Lambda}. \tag{6.14}
$$

Thus, the matrices $\boldsymbol{C}^{(\pm)}$ arise here as well. Closure of the algebra reduces to the condition

$$
[\delta_1, \delta_2]\mathbf{\Lambda} = 2\epsilon_1^+ \epsilon_2^+ \partial_{+\!\!+} \mathbf{\Lambda} + 2\epsilon_1^- \epsilon_2^- \partial_{=} \mathbf{\Lambda}. \tag{6.15}
$$

It is not difficult to check that this operator formulation works also for (ordinary) second order sigma models and in the context of generalized complex geometry on $TM \oplus T^*M$.

7. Manifest supersymmetry and left-/right-chiral superfields

There are several ways to construct manifest $N = (2, 2)$ sigma models by using constrained $N = (2, 2)$ superfields. The different possibilities are chiral, twisted chiral and left-/right-chiral ones together with their antichiral partners [8, 15, 26, 27, 28]. To understand how the latter, originally called semichiral fields, fit into our description in terms of $N = (1, 1)$ manifest supersymmetry, we start with the simple toy model action

$$
S = -\int \mathrm{d}^2\xi \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \Big[\mathbb{X}\bar{\mathbb{Y}} - \bar{\mathbb{X}}\mathbb{Y}\Big] = \int \mathrm{d}^2\xi \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \Big[\mathbb{X}^A B_{AB'} \mathbb{Y}^{B'}\Big] \tag{7.1}
$$

$\mathbb{X}, \mathbb{Y}$ are the left-chiral and right-antichiral superfields [14]:

$$\begin{aligned} &\bar{\mathbb{D}}_+ \mathbb{X} = 0 && \mathbb{D}_- \mathbb{Y} = 0 \\ &D_\pm = \mathbb{D}_\pm + \bar{\mathbb{D}}_\pm && Q_\pm = \mathrm{i}(\mathbb{D}_\pm - \bar{\mathbb{D}}_\pm) \\ &\varphi = \mathbb{X}| && \Psi_- = Q_- \mathbb{X}| \\ &\chi = \mathbb{Y}| && \Upsilon_+ = Q_+ \mathbb{Y}|. \end{aligned} \tag{7.2}$$

The p left chiral and p' right antichiral holomorphic coordinate indices a and a' and their antiholomorphic partners are conveniently collectively denoted $A = a, \bar{a}$ and $A' = a', \bar{a}'$. Moreover, we introduce $\alpha = A, A'$.

After a redefinition of the fields,

$$\begin{aligned} (\Psi_-, \bar{\Upsilon}_+) &\to (\hat{\Psi}_-, \hat{\bar{\Upsilon}}_+) = (\Psi_-, \bar{\Upsilon}_+) - \mathrm{i}(D_-\varphi, D_+\bar{\chi}) \\ (\bar{\Psi}_-, \Upsilon_+) &\to (\hat{\bar{\Psi}}_-, \hat{\Upsilon}_+) = (\bar{\Psi}_-, \Upsilon_+) + \mathrm{i}(D_-\bar{\varphi}, D_+\chi) \end{aligned} \tag{7.3}$$

and with $B_{A'B} \equiv -B_{BA'}$, the action (7.1) becomes

$$S = -\int \mathrm{d}^2\xi \mathrm{d}^2\theta \Big[\hat{\Psi}_-^A B_{AB'} \hat{\Upsilon}_+^{B'} + D_+\varphi^A B_{AB'} D_-\chi^{B'} + D_+\chi^{A'} B_{A'B} D_-\varphi^B \Big]. \tag{7.4}$$

We find it convenient to collect the fields into

$$\phi^\alpha = (\varphi^A, \chi^{A'}) \qquad \Psi_+^\alpha = (\hat{\Psi}_+^A, \hat{\Upsilon}_+^{A'}) \qquad \Psi_-^\alpha = (\hat{\Psi}_-^A, \hat{\Upsilon}_-^{A'}) \tag{7.5}$$

and introduce

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B_{AB'} \\ B_{A'B} & 0 \end{pmatrix}. \tag{7.6}$$

We let $S_{+\alpha} = \Psi_+^\kappa B_{\kappa\alpha}$ and $S_{-\alpha} = B_{\alpha\kappa}\Psi_-^\kappa$ and denote the inverse of $B_{\alpha\beta}$ by $\Pi^{\alpha\beta}$. We may then rewrite (7.1) as

$$S = -\int \mathrm{d}^2\xi \mathrm{d}^2\theta \Big[S_{+\alpha}\Pi^{\alpha\beta}S_{-\beta} + D_+\phi^\alpha B_{\alpha\beta} D_-\phi^\beta \Big] \tag{7.7}$$

where Π and B are constant antisymmetric tensors by definition. This implies that the second term vanishes, however, we keep it for clarity. Even though $\Pi = B^{-1}$, we distinguish them to keep as close as possible to the discussion in the previous sections. The fields $S_\pm$ are constrained by

$$S_{-A} = S_{+A'} = 0 \qquad\qquad \hat{\Upsilon}_-^{A'} = \hat{\Psi}_+^A = 0. \tag{7.8}$$

The $2p + 2p'$ constraints on the $N = (2,2)$ fields (7.2) translate into restrictions on the auxiliary fields $S_\pm$. Effectively, half of them have been integrated out by means

of their field equations (7.8). This is the direct translation of the constraints (7.2) on the $N = (2,2)$ fields. We may formally introduce

$$\Psi_+ = Q_+\mathbb{X}| \qquad \Upsilon_- = Q_-\mathbb{Y}| \qquad \bar{\Psi}_+ = Q_+\bar{\mathbb{X}}| \qquad \bar{\Upsilon}_- = Q_-\bar{\mathbb{Y}}|. \tag{7.9}$$

The constraints (7.2) transform into

$$\Psi_+ = \mathrm{i}D_+\varphi \qquad \Upsilon_- = -\mathrm{i}D_-\chi \qquad \bar{\Psi}_+ = -\mathrm{i}D_+\bar{\varphi} \qquad \bar{\Upsilon}_- = \mathrm{i}D_-\bar{\chi}. \tag{7.10}$$

If we define the transformation (7.3) on these fields by

$$\begin{aligned} (\Psi_+, \bar{\Upsilon}_-) &\to (\hat{\Psi}_+, \hat{\bar{\Upsilon}}_-) = (\Psi_+, \bar{\Upsilon}_-) - \mathrm{i}(D_+\varphi, D_-\bar{\chi}) \\ (\bar{\Psi}_+, \Upsilon_-) &\to (\hat{\bar{\Psi}}_+, \hat{\Upsilon}_-) = (\bar{\Psi}_+, \Upsilon_-) + \mathrm{i}(D_+\bar{\varphi}, D_-\chi) \end{aligned} \tag{7.11}$$

we find that (7.8) and (7.10) match each other. Thus, using the field equations for half of the auxiliary $N = (1,1)$ fields is equivalent to constraining the $N = (2,2)$ superfields. By a simple rotation we see that we cover all cases which allow for Darboux-Nijenhuis coordinates, even for non-constant B.

By integrating out some of the fields, the almost complex structure matrices effectively collapse to generalized complex structures:

$$\boldsymbol{J}^{(+)} = \begin{pmatrix} J^{(+)} & -P^{(+)} & 0 \\ 0 & K^{(+)} & 0 \\ 0 & -Z^{(+)} & R^{(+)} \end{pmatrix} \longrightarrow \mathcal{J}^{(+)} = \begin{pmatrix} \hat{J}^{(+)} & -\hat{P}^{(+)} \\ 0 & \hat{K}^{(+)} \end{pmatrix}. \tag{7.12}$$

In terms of A, A' coordinates, this reads

$$\mathcal{J}^{(+)} = \left(\begin{array}{cc|cc} J^{(+)} & 0 & 0 & 0 \\ 0 & J^{(+)\prime} & -P^{(+)} & 0 \\ \hline 0 & 0 & K^{(+)} & 0 \\ 0 & 0 & -Z^{(+)} & R^{(+)\prime} \end{array}\right) \tag{7.13}$$

where we identified the tensors with their remaining components, e.g. $K^\alpha_\beta \to K^A_B$. There is a similar reduction for $\boldsymbol{J}^{(-)} \to \mathcal{J}^{(-)}$. Comparing to the results of [15], we find their solution to match ours, (7.13) if $Z^{(\pm)} = 0$. We find $P^{(\pm)} = -P^{(\mp)t}$. It follows $J^{(+)} = -J^{(-)}$ and $R^{(\pm)} = -K^{(\pm)}$. $Z^{(\pm)} = 0$ implies $[J^{(\pm)}, \omega] = 0$, with ω as defined in section 3 where we identified this case with a symplectic manifold.

In the non-manifest description, we found a whole set of solutions given in terms of almost complex structure matrices to choose among. In the left-/right-chiral description, we start with ‘diagonal’ objects, i.e. Darboux-Nijenhuis coordinates. We learn that the different alternatives should collapse into one and the same in (reduced) Darboux-Nijenhuis coordinates. This implies that we can choose $Z^{(\pm)} = 0$ in these cases.

If we integrate out the remaining spinorial fields we reduce the geometry to the ordinary case of two (ordinary, commuting) complex structures $J^{(\pm)}$ acting on TM. We obtain the following diagram:

$$\boldsymbol{J}^{(\pm)} \xrightarrow{S_{+A'}, S_{-A}=0} \mathcal{J}^{(\pm)} \xrightarrow{S_{+A}, S_{-A'}=0} J^{(\pm)}$$

We find the line of argumentation valid even when replacing $B_{AB'}$ by a non-constant $E = G + B$ in (7.1), following the lines of [15]. This strongly suggests that this is the general way to understand and embed left-/right-(anti-)chiral $N = (2, 2)$ theories in this context.

8. Discussion

We presented a new framework that might lead to new insights on the way towards a complete understanding of the geometry underlying $N = (2, 2)$ supersymmetric non-linear sigma models. As examples we considered the symplectic sigma model and showed how the manifest theories in terms of left-/right-chiral superfields can be interpreted and embedded into our new framework.

It is an open question, how to treat arbitrary first order sigma models in this context. Most intriguing is the question what the proper integrability conditions are, mainly due to the lack of a proper language. One might expect integrability to work out in a similar way as Courant integrability generalizes ordinary integrability. For the particular models we studied, we found a description that is given by a covariantly constancy condition of the almost complex structure matrices. We argued that this gives the integrability condition for these models. The solution is based on the invertibility of one of the complex structure tensors. We elaborated on possible generalizations of our solution and compared them to the corresponding second order sigma model. The most general target space geometry allowing for the extension of a general $N = (1, 1)$ sigma model to $N = (2, 2)$ supersymmetry is still to be discovered, although we believe that that framework presented here contributes an important step in that direction.

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